TRansforming Instruction in Undergraduate Mathematics with Primary Historical Sources

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Benefits to Using Primary Sources in Learning

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- They engender cognitive dissonance (dépaysement) by comparison of a historical source with a modern approach
- They promote understanding of present-day paradigms through source texts which require no knowledge of that paradigm
Active Learning Principles

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TRIUMPHS

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- Large group work: instructor leads whole-classroom discussions that clarifies ideas and explains subtleties.
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• Lecture is NOT abandoned, but takes a smaller role
Challenges to this Approach

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- Active learning requires engagement on the part of students during the classroom hour – with the PSP, with their classmates, with the instructor, and not all students are used to this …especially the most successful ones!
a mini-PSP

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- Used in history of maths courses, and courses treating foundations of arithmetic for school teachers
Babylonian Numeration

Task 1: How do Babylonian numerals work?

Task 2: Describe the mathematics on this tablet.

Task 3: Write the number 72 in Babylonian numerals.
Babylonian Numeration

The next 30 minutes

- Cries of “impossible!” – followed by quiet, confused murmurs – cautious suggestions – then strong declarations of conviction about what is going on – then revisions of these declarations
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- Cautious conclusions with help from other small groups
- Finally, an instructor-led full-class discussion about the mathematics, and about the benefits and drawbacks of Tasks of this sort.
A Genetic Context for Understanding the Trig Functions
Six episodes in the history of trigonometry.
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* [early 2nd millennium BCE] **Babylonian numeration**
...Thus, $2\frac{3}{4}$ is written in sexagesimal form as $[2; 45]$, 78 is written as $[1, 18]$, 4.0075 is written as $[4; 0, 27]$, and 4800 is written as $[1, 20, 0]$.

**Task 1:** Verify that the numbers 78, 4.0075 and 4800 have the sexagesimal representations identified above. Explain how you know that these representations are correct.
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* [ca. 130 BCE] A reconstruction of Hipparchus’ table of chords following classical Greek methods
**Hipparchus’ table of chords**

![Hipparchus' Table of Chords](image)

**Figure 1: Hipparchus’ Table of Chords (Reconstruction)**

<table>
<thead>
<tr>
<th>Arcs</th>
<th>Chords</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7, 30]</td>
<td>[7, 30]</td>
</tr>
<tr>
<td>[15, 0]</td>
<td>[14, 57]</td>
</tr>
<tr>
<td>[22, 30]</td>
<td>[22, 21]</td>
</tr>
<tr>
<td>[30, 0]</td>
<td>[29, 40]</td>
</tr>
<tr>
<td>[37, 30]</td>
<td>[36, 50]</td>
</tr>
<tr>
<td>[45, 0]</td>
<td>[43, 51]</td>
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<tr>
<td>[52, 30]</td>
<td>[50, 41]</td>
</tr>
<tr>
<td>[60, 0]</td>
<td>[57, 18]</td>
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<tr>
<td>[67, 30]</td>
<td>[63, 40]</td>
</tr>
<tr>
<td>[75, 0]</td>
<td>[69, 46]</td>
</tr>
<tr>
<td>[82, 30]</td>
<td>[75, 33]</td>
</tr>
<tr>
<td>[90, 0]</td>
<td>[81, 2]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Arcs</th>
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</tr>
</thead>
<tbody>
<tr>
<td>[97, 30]</td>
<td>[86, 9]</td>
</tr>
<tr>
<td>[105, 0]</td>
<td>[90, 55]</td>
</tr>
<tr>
<td>[112, 30]</td>
<td>[95, 17]</td>
</tr>
<tr>
<td>[120, 0]</td>
<td>[99, 14]</td>
</tr>
<tr>
<td>[127, 30]</td>
<td>[102, 46]</td>
</tr>
<tr>
<td>[135, 0]</td>
<td>[105, 52]</td>
</tr>
<tr>
<td>[142, 30]</td>
<td>[108, 31]</td>
</tr>
<tr>
<td>[150, 0]</td>
<td>[110, 41]</td>
</tr>
<tr>
<td>[157, 30]</td>
<td>[112, 23]</td>
</tr>
<tr>
<td>[165, 0]</td>
<td>[113, 37]</td>
</tr>
<tr>
<td>[172, 30]</td>
<td>[114, 21]</td>
</tr>
<tr>
<td>[180, 0]</td>
<td>[114, 35]</td>
</tr>
</tbody>
</table>

**Task 7(b):** Crd[60; 0] should be equal to the radius of the underlying circle. Show how Hipparchus’ Table verifies this.
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Ptolemy’s *Almagest*

Then, since arc GD, which is equal to the elevation of the north pole from the horizon, is $36^\circ$... at the latitude in question, and both arc $\Theta D$ and arc DM are $[23; 51, 20]^\circ$, by subtraction arc $G\Theta = [12; 8, 40]^\circ$, and by addition arc GM = $[59; 51, 20]^\circ$.

Therefore the corresponding angles

$\angle KEG = [12; 8, 40]^\circ$,

$\angle ZEG = 36^\circ$,

$\angle NEG = [59; 51, 20]^\circ$
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A sine table in Sanskrit verse

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6 The Sines in Aries are 7, 15, 20 plus 3, plus 11, and plus 18, 45, 50 plus 3, and 60 minutes;
7 in Aries 50 plus 1, 5 times 8, 5 squared, 4, 30 plus 4, 56, 5, and 0 [seconds].
8 In Taurus [they are] 6, 13, 19, 3 times 8, and 30 – plus 0, plus 5, plus 9, and plus 13 minutes;
in Taurus 40, 3, 7, 50 plus 1, 13, 12, and 60 minus 14 and minus 5 seconds.

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Task 22(b). Thus, the first entry gives the Sine of 3°45′ as 7 minutes, 50 + 1 = 51 seconds (or 7′51″). Varahamihira uses odd language to fit the poetic meter of the Sanskrit, a sense that is entirely lost in English translation! Now fill in the rest of your table with the Sine values reported in the text.
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* [1021 CE] From al-Bīrūnī’s *Exhaustive Treatise on Shadows*, measuring shadows lead to tabulations of “tangent, cotangent” and “secant, cosecant”
An example of the *direct shadow* is [the following (see the figure)]: Let A be the body of the sun and BG the gnomon perpendicular to EG, which is parallel to the horizon plane, and ABE is the sun's ray passing through the head of the gnomon BG. So will BGE be the shadow in space. But EG is that which is called the direct shadow such that its base is G and its end E. And EB, the line joining the two ends of the shadow and the gnomon, is the *hypotenuse of the shadow*.
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* [1533 CE] **Unified mathematical approach to trigonometry in Regiomontanus’ *On Triangles of All Kinds***
Theorem 29. When one of the two acute angles and one side of a right triangle are known, all the angles and sides may be found. [...] 

For instance, let angle ABC be given as 36° and side AB as 20 feet. Subtract 36 from 90 to get 54°, the size of angle BAC. Moreover, from the table of sines it is found that line AC is 35267 while BC is 48541 (when AB, the whole sine, is 60000). Therefore, multiplying 35267 by 20 yields 705340, which when divided by 60000, leaves about $11 \frac{45}{60}$. Thus, side AC will have 11 feet and $\frac{45}{60}$ – that is, three-fourths – of one foot. Similarly multiply 48541 by 20, giving 970820, which when divided by 60000, leaves about 16 [feet] and 11 minutes,29 the length of side BC.

**Task 34(a).** Draw the triangle, labeling the known angles and the hypotenuse as 20 feet. Use Regiomontanus’ sine values and appropriate proportions to find his lengths $AC = 11 \frac{45}{60}$ and $BC = 16 \frac{11}{60}$. 
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- Only a partial implementation, completing 15 of 34 Tasks
- Materials (32 pages) assigned for reading before first class
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- Traditional trigonometry module followed on (but was much abbreviated)
- Formerly least engaged students became most interested and engaged during the PSP!
Other Projects I Have Used

• **Geometry:** *The Exigency of the Parallel Postulate* gives Euclid’s proof of the Pythagorean Theorem and analyzes its dependence on the Parallel Postulate
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- **Group Theory**: *Otto Hölder’s Formal Christening of the Quotient Group Concept* introduces normal subgroups, quotient groups and the First Homomorphism Theorem.
Other Available Projects

There are 49 PSPs currently available (more planned), free for download, which present topics in courses on

- Pre-calculus and trigonometry
- Calculus
- Statistics and probability
- Discrete mathematics
- Differential equations
- Geometry
- Complex numbers
- Linear algebra
- Number theory
- Abstract algebra
- Analysis
- Topology

Visit https://blogs.ursinus.edu/triumphs/
or Google "TRIUMPHS mathematics"
Thanks for your attention!

Questions?